The Frölicher – Kriegl differentiabilities as a particular case of the Bertram – Glöckner – Neeb construction

by

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Abstract. We prove that the order k differentiability classes for $k = 0., 1., \ldots \infty$ in the "arc-generated" interpretation of the Lipschitz theory of differentiation by Frölicher and Kriegl can be obtained as particular cases of the general construction by Bertram, Glöckner and Neeb leading to C^k differentiabilities from a given C^0 concept.

In [2; p. 252], it was already announced (without proof) that the general construction in [2] gives as particular cases the Lipschitz differentiabilities of order k in [3] for maps $f : E \supseteq U \to F$ when the spaces E, F are equipped with the c^{∞} – extensions of the locally convex topologies. Our purpose in this note is to prove this result, formulated as Theorem 12 below. This complements our treatment in [4]. For most of the notations and of the preliminaries, we refer to [4; pp. 4 – 9] which we assume the reader to be acquainted with. However, we introduce the following slight change in notation.

Below, the standard topological field of real numbers is ${}^{\mathrm{tf}}\mathbb{R} = ({}^{\mathrm{f}}\mathbb{R}, \tau_{\mathbb{R}})$ with underlying set $\mathbb{R}$ instead of the earlier $\boldsymbol{R} = (\mathbf{R}, \tau_{I\!R})$ over $I\!R$. We also let $\mathbb{R}^+ = \mathbb{R} \cap \{ t : t > 0 \}$, and $\mathbb{N}_0 = \infty$ is the set of natural numbers. Further, we define

$$[f, g]_{\mathrm{f}} = \{ (x ; y, z) : (x, y) \in f \text{ and } (x, z) \in g \} \qquad \text{and}$$
$$f \underset{\mathrm{f}}{\times} g = \{ (x, u ; y, v) : (x, y) \in f \text{ and } (u, v) \in g \}$$

which earlier were written "$[f, g]$" and "$f \underset{2}{\times} g$", respectively.

In general definitions, to fix matters precisely, we utilize the notational convention for linear combinations sketched in [4; p. 5] according to which for example we have $(t\,x + s\,y)_{\mathrm{svs}\,E} = \sigma_{\mathrm{rd}}^2 E\,\grave{}\,(\tau\sigma_{\mathrm{rd}} E\,\grave{}\,(t, x), \tau\sigma_{\mathrm{rd}} E\,\grave{}\,(s, y))$ which more shortly and generally ambiguously is denoted by "$t\,x + s\,y$" when E is a real structured vector space and we have $s, t \in \mathbb{R}$ and $x, y \in v_s E$. Likewise, for example instead of the precise $[\{x\} + \{\delta\}\, B]_{\mathrm{svs}\,E}$ one conventionally writes "$x + \delta\, B$". However, in passages of informal discussion or in proofs where the surrounding spaces have been fixed, we may use the shorter imprecise notations.

By a *structure changer* meaning any function $\sigma \subseteq (\mathbf{U}^{\times 2.})^{\times 2.} = \mathbf{U} \times \mathbf{U} \times (\mathbf{U} \times \mathbf{U})$ with $\mathrm{pr}_1 \circ \sigma \subseteq \mathrm{pr}_1$, the *interpretation* of a class $\mathcal{C}$ of $\mathbf{K}$ – vector maps by a structure changer σ satisfying also $\mathrm{dom}^2 \mathcal{C} \cup \mathrm{rng}\,\mathrm{dom}\,\mathcal{C} \subseteq \mathrm{dom}\,\sigma$ we understand the class $\sigma \underset{\mathrm{f}}{\times} \sigma \underset{\mathrm{f}}{\times} \mathrm{id}\,\grave{}\grave{}\,\mathcal{C} = \{ (\sigma\,\grave{}\,E, \sigma\,\grave{}\,F, f) : (E, F, f) \in \mathcal{C} \}$. In [3] structure changers frequently appear as object components of functors.

We next suitably reformulate the facts from [3] which we need below.

2000 *Mathematics Subject Classification.* Primary 46T20, 46G05; Secondary 58C25, 58C20, 46A17.

Key words and phrases. Differentiability, Frölicher – Kriegl theory, BGN – theory, arc-generated /determined vector space, Lipschitz map, dualized vector space, locally convex space.

A. The convenient spaces of Frölicher and Kriegl

Let $\mathrm{strVS}({}^{\mathrm{f}}\mathbb{R}) = \{(X,S) : X \text{ real vector space}\}$, the class of all structured real vector spaces. Its subclass $\mathrm{LCS}({}^{\mathrm{tf}}\mathbb{R})$ has only a minor role below. The subclasses AVS and DVS will be more important here. They are obtained as follows.

Put $\mathrm{DVS} = \mathrm{strVS}({}^{\mathrm{f}}\mathbb{R}) \cap \{E : (E)_{\mathrm{D}}\}$ where $(E)_{\mathrm{D}}$ means that $\tau_{\mathrm{rd}} E \neq \emptyset$ is a point separating linearly closed set of linear maps $\sigma_{\mathrm{rd}} E \to {}^{\mathrm{f}}\mathbb{R}$. Calling a topology $\mathcal{T}$ *arc-generated* iff $U \in \mathcal{T}$ for every $U \subseteq \bigcup \mathcal{T}$ with $c^{-\iota}``U \in \tau_{\mathbb{R}}$ for all continuous $c : \tau_{\mathbb{R}} \to \mathcal{T}$ having $\mathrm{dom}\, c = \mathbb{R}$, we define $\mathrm{AVS} = \mathrm{strVS}({}^{\mathrm{f}}\mathbb{R}) \cap \{E : (E)_{\mathrm{A}}\}$ where $(E)_{\mathrm{A}}$ means that $\tau_{\mathrm{rd}} E$ is an arc-generated Hausdorff topology for $\upsilon_s E$ such that for $\mathcal{P} = (\tau_{\mathbb{R}}, \tau_{\mathrm{rd}} E)$ we have $(\mathcal{P}, \sigma_{\mathrm{rd}}^2 E \circ [c_1, c_2]_{\mathrm{f}})$ and $(\mathcal{P}, \tau\sigma_{\mathrm{rd}} E \circ [c, c_1]_{\mathrm{f}})$ topological maps when $(\tau_{\mathbb{R}}, \tau_{\mathbb{R}}, c)$ and $(\mathcal{P}, c_\iota)$ are global topological maps for ${}_{\iota = 1,2}$.

The reader may compare the preceding definitions to [3; Definition 2.1.1, Remark, p. 28, Definitions 2.3.8, 2.3.9, p. 45]. In particular, note that we make all the spaces "separated" right from the beginning, as opposed to [3]. By a simple verification $E \in \mathrm{AVS}$ when $E \in \mathrm{TVS}({}^{\mathrm{tf}}\mathbb{R})$ with $\tau_{\mathrm{rd}} E$ a metrizable topology.

We say that $\mathcal{T}$ is the *smooth S-topology* iff there is Ω with $\emptyset \neq S \subseteq \mathbb{R}^{\Omega}$ and

$$\mathcal{T} = \{U : U \subseteq \Omega \text{ and } \forall\, c\,;\, c \in \Omega^{\mathbb{R}} \text{ and } [\,\forall \varphi \in S\,;\, \varphi \circ c \in \upsilon_s C^{\infty}(\mathbb{R})\,] \Rightarrow c^{-\iota}``U \in \tau_{\mathbb{R}}\}.$$

Then putting

$$\delta_{\mathrm{av}} = \langle (\sigma_{\mathrm{rd}} E, \mathcal{L}(E, {}^{\mathrm{tf}}\mathbb{R})) : E \in \mathrm{AVS} \rangle \quad \text{and}$$

$$\tau_{\mathrm{dv}} = \langle (\sigma_{\mathrm{rd}} E, \mathcal{T}) : E \in \mathrm{DVS} \text{ and } \mathcal{T} \text{ is the smooth } \tau_{\mathrm{rd}} E\text{-topology} \rangle,$$

we get the structure changers $\tau_{\mathrm{dv}} : \mathrm{DVS} \to \mathrm{AVS}$ and $\delta_{\mathrm{av}} : \mathrm{AVS} \to \mathrm{strVS}({}^{\mathrm{tf}}\mathbb{R})$. Here $\tau_{\mathrm{dv}} = \tau_{\mathrm{M}} \,|\, \mathrm{DVS}$ when τ_{M} is the object component of the functor

$$\boldsymbol{\tau}_{\mathrm{M}} : \underline{\mathrm{DVS}}_{\mathrm{there}} \to \underline{\mathrm{ArcVS}}_{\mathrm{there}} \text{ in } [3; \text{Definition 2.3.14, p. 46}].$$

Taking for example $E = L^{\frac{1}{2}}([0,1])$, we have $\tau_{\mathrm{rd}}(\delta_{\mathrm{av}}`E) = \{\upsilon_s E \times \{0\}\}$, hence $\delta_{\mathrm{av}}`E \notin \mathrm{DVS}$, and consequently $\mathrm{rng}\, \delta_{\mathrm{av}} \not\subseteq \mathrm{DVS}$.

Now, the class of *dualized Frölicher–Kriegl convenient* spaces is

$$\mathrm{Con}_{\mathrm{FKd}} = \mathrm{DVS} \cap \{E : \mathcal{L}(\tau_{\mathrm{dv}}`E, {}^{\mathrm{tf}}\mathbb{R}) \subseteq \tau_{\mathrm{rd}} E \text{ and } (E)_{\mathrm{C}}\}$$

where $(E)_{\mathrm{C}}$ means that for every $c \in (\upsilon_s E)^{\mathbb{R}}$ there is some $c_1 \in (\upsilon_s E)^{\mathbb{R}}$ such that if $\ell \circ c \in \upsilon_s C^{\infty}(\mathbb{R})$ holds for all $\ell \in \tau_{\mathrm{rd}} E$, then we also have $\ell \circ c_1 = (\ell \circ c)'$ for $\ell \in \tau_{\mathrm{rd}} E$. The reader may compare this to [3; Definitions 2.4.2, 2.5.3, 2.6.3, pp. 48, 53, 57]. Putting $\mathrm{Con}_{\mathrm{FKa}} = \tau_{\mathrm{dv}}``\mathrm{Con}_{\mathrm{FKd}}$, then $\mathrm{Con}_{\mathrm{FKa}}$ is the *arc-generated counterpart* of the class of Frölicher–Kriegl convenient spaces, and essentially from [3; Theorems 2.4.3(vi), 2.5.2, pp. 49, 53] we obtain the following

1 Lemma. $\tau_{\mathrm{dv}} \,|\, \mathrm{Con}_{\mathrm{FKd}}$ *is bijective* $\mathrm{Con}_{\mathrm{FKd}} \to \mathrm{Con}_{\mathrm{FKa}}$ *and* $(\tau_{\mathrm{dv}} \,|\, \mathrm{Con}_{\mathrm{FKd}})^{-\iota} = \delta_{\mathrm{av}} \,|\, \mathrm{Con}_{\mathrm{FKa}}$.

Letting $\mathrm{Con}_{\mathrm{KM}} = \{E : (E)_{\mathrm{con\,KM}}\}$, where $(E)_{\mathrm{con\,KM}}$ means that $E \in \mathrm{LCS}({}^{\mathrm{tf}}\mathbb{R})$ and is locally/Mackey complete in the sense [5; p. 196] or [6; Lemma 2.2, p. 15], then $\mathrm{Con}_{\mathrm{KM}}$ is the class of spaces which are "convenient" in the sense [6; Theorem 2.14, p. 20]. The class $\mathrm{Con}_{\mathrm{FKt}} = \{E : (E)_{\mathrm{con\,KM}} \text{ and } E \text{ is bornological}\}$ is the *locally convex counterpart* of the class of Frölicher–Kriegl convenient spaces.

For $E \in \mathrm{LCS}({}^{\mathrm{tf}}\mathbb{R})$, let $\tau_{\mathrm{Mac}} E = \{U : U \text{ mopen in } E\}$ where U being *mopen* in E means that $U \subseteq \upsilon_s E$ and that for $x \in U$ and $B \in \mathcal{B}_s E$ there is some $\delta \in \mathbb{R}^+$ with $[\{x\} + \{\delta\} B]_{\mathrm{svs}\, E} \subseteq U$. Then $\tau_{\mathrm{Mac}} E$ is the *Mackey closure* topology of the *bornological* vector space $(\sigma_{\mathrm{rd}} E, \mathcal{B}_s E)$ which in [6; Definition 2.12, p. 19] is also called the c^{∞}-*topology* of the *locally convex* space E.

Considering a fixed $E \in \mathrm{Con}_{\mathrm{FKd}}$, when one a bit loosely speaks of the "Mackey closure topology" in various places in [3] without explicitly specifying the bornological vector space, which there would be denoted by "$\sigma_{\mathrm{b}} E$", note that by [3; Proposition 2.3.7, p. 44] one then refers precily to the topology $\tau_{\mathrm{rd}}(\tau_{\mathrm{dv}}{}^{\backprime}E)$. This should be kept in mind when we below refer to results in [3] in order to get some shortening of presentation.

To summarize, if we put $\alpha_{\mathrm{FKt}} = \langle (\sigma_{\mathrm{rd}} E, \tau_{\mathrm{Mac}} E) : E \in \mathrm{Con}_{\mathrm{FKt}} \rangle$ and $\delta_{\mathrm{FKa}} = \delta_{\mathrm{av}} | \mathrm{Con}_{\mathrm{FKa}}$, we have

$$\begin{tikzcd}
\mathrm{AVS} \supset \mathrm{Con}_{\mathrm{FKa}} \arrow[d, "\delta_{\mathrm{FKa}}"] & \mathrm{Con}_{\mathrm{FKt}} \subset \mathrm{Con}_{\mathrm{KM}} \subset \mathrm{LCS}({}^{\mathrm{tf}}\mathbb{R}) \arrow[l, "\alpha_{\mathrm{FKt}}"'] \\
\mathrm{DVS} \supset \mathrm{Con}_{\mathrm{FKd}} &
\end{tikzcd}$$

where α_{FKt} and δ_{FKa} are bijective. This diagram is here given only for the purpose of clarifying the relations between the various classes of structured vector spaces. Below, we only need to consider the bijection given by the vertical arrow.

2 *Definitions* (product structures)**.**

$$\begin{aligned}
X \underset{\mathrm{vs}}{\times} Y = \textstyle\bigcap \{ (a, c) : \exists\, a_1, a_2, c_1, c_2 \in \mathbf{U}\,;\ & X = (a_1, c_1) \text{ and } Y = (a_2, c_2) \\
\text{and } a = \{ (x, y; u, v; z, w) : {} & (x, u, z) \in a_1 \text{ and } (y, v, w) \in a_2 \} \\
\text{and } \quad c = \{ (t; x, y; u, v) : {} & (t, x, u) \in c_1 \text{ and } (t, y, v) \in c_2 \} \} ,
\end{aligned}$$

$$\mathcal{T} \underset{\mathrm{t}}{\times} \mathcal{U} = \{ \textstyle\bigcup \mathcal{A} : \mathcal{A} \subseteq \{ U \times V : U \in \mathcal{T} \text{ and } V \in \mathcal{U} \} \} ,$$

$$E \sqcap F = ((\sigma_{\mathrm{rd}} E) \underset{\mathrm{vs}}{\times} (\sigma_{\mathrm{rd}} F), (\tau_{\mathrm{rd}} E) \underset{\mathrm{t}}{\times} (\tau_{\mathrm{rd}} F)) ,$$

$$\begin{aligned}
E \boxed{\scriptstyle\mathrm{d}} F = \textstyle\bigcap \{ (X, S) : X = (\sigma_{\mathrm{rd}} E) \underset{\mathrm{vs}}{\times} (\sigma_{\mathrm{rd}} F) \text{ and } S = \{ \ell : \exists\, \ell_1 \in \tau_{\mathrm{rd}} E ,& \\
\ell_2 \in \tau_{\mathrm{rd}} F\,;\ \ell = \{ (x, y, r + s) : (x, r) \in \ell_1 \text{ and } (y, s) \in \ell_2 \} \} \} ,&
\end{aligned}$$

$$E \boxed{\scriptstyle\mathrm{a}} F = \tau_{\mathrm{dv}}{}^{\backprime}((\delta_{\mathrm{av}}{}^{\backprime}E) \boxed{\scriptstyle\mathrm{d}} (\delta_{\mathrm{av}}{}^{\backprime}F)) .$$

We may call $X \underset{\mathrm{vs}}{\times} Y$ the *vector space* (or module) product of X and Y. The class $\mathcal{T} \underset{\mathrm{t}}{\times} \mathcal{U}$ is the Tihonov topological product of $\mathcal{T}$ and $\mathcal{U}$.

By [3; Proposition 3.3.1, p. 65], when $E, F \in \mathrm{Con}_{\mathrm{FKd}}$, we have that $E \boxed{\scriptstyle\mathrm{d}} F$ is precisely the product space which in [3] is denoted by "$E \,\Pi\, F$". In particular, recalling Lemma 1 above, we have that $E, F \in \mathrm{Con}_{\mathrm{FKd}} \Rightarrow E \boxed{\scriptstyle\mathrm{d}} F \in \mathrm{Con}_{\mathrm{FKd}}$ and that $E, F \in \mathrm{Con}_{\mathrm{FKa}} \Rightarrow E \boxed{\scriptstyle\mathrm{a}} F \in \mathrm{Con}_{\mathrm{FKa}}$. By [3; Remark 3.3.4, p. 67], we have $E \boxed{\scriptstyle\mathrm{a}} F = E \sqcap F$ whenever the spaces $E, F \in \mathrm{Con}_{\mathrm{FKa}}$ are such that at least one of them is finite-dimensional.

B. The Lipschitz differentiable maps

We refer to (5) and (6) and (7) of Constructions 3 below. In [3; p. 83], it is agreed that $f : E \supseteq U \to F$ is $\mathcal{L}ip^{\,k}$ iff $E, F \in \mathrm{Con}_{\mathrm{FKd}}$ and $U \in \tau_{\mathrm{rd}}(\tau_{\mathrm{dv}}{}^{\backprime}E)$ and $f \in (v_s F)^U$ with $f \circ c \in \mathcal{L}ip^{\,k}_{\mathbb{R}} F$ for $c \in \mathcal{L}ip^{\,k}_{\mathbb{R}} E$ having $\operatorname{rng} c \subseteq U$. Assuming that $k \in \infty^+$, and also taking [3; Definition 1.4.1, Proposition 2.3.7, Lemma 4.3.1, pp. 22, 44, 99] into account, it is seen that $f : E \supseteq U \to F$ is $\mathcal{L}ip^{\,k}$ if and only if $(E, F, f) \in \mathcal{L}ip^{\,k}_{\mathrm{FKd}}$ with $\operatorname{dom} f = U$. Hence, for $k \in \infty^+$ we may say that $\mathcal{L}ip^{\,k}_{\mathrm{FKd}}$ is the class of maps which are Lipschitz differentiable of order k in the sense of Frölicher and Kriegl, and that $\mathcal{L}ip^{\,k}_{\mathrm{FKa}}$ is its "arc-generated" interpretation.

3 *Constructions* (classes of Lipschitz functions and maps)**.** For all E, k, with the restriction $E \in \mathrm{DVS}$ in (4) and (5) below, we let

(1) $\mathcal{L}ip = \{\, \chi : \exists\, \Omega \in \tau_{\mathbb{R}}\, ;\, \chi \in \mathbb{R}^{\,\Omega}$ and $\forall\, t_0 \in \Omega\, ;\, \exists\, \delta \in \mathbb{R}^+\, ;$
$\forall\, s, t \in \Omega\, ;\, |\, s - t_0 \,| + |\, t - t_0 \,| < \delta \Rightarrow |\, \chi`s - \chi`t \,| \le \delta^{-1} |\, s - t \,| \,\}$,

(2) $\mathcal{L}ip^{\,k} = \mathcal{L}ip \cap \big\{\, \chi : k \in \infty^+$ and $\exists\, \Omega, \boldsymbol{\chi}\, ;\, (\emptyset, \chi) \in \boldsymbol{\chi} \in \big(\mathbb{R}^{\,\Omega}\big)^{\,k+1.}$
and $\forall\, i \in k\, ;\, \chi`i^+ = (\chi`i)' \in \mathcal{L}ip \,\big\}$,

(3) $\mathcal{L}ip^{\,k}_{\,\mathbb{R}} = \mathcal{L}ip^{\,k} \cap \mathbb{R}^{\,\mathbb{R}}$,

(4) $\mathcal{L}ip^{\,k} E = \{\, c : c \in (\upsilon_s E)^{\,\mathrm{dom}\, c}$ and $\forall\, \ell \in \tau_{\mathrm{rd}} E\, ;\, \ell \circ c \in \mathcal{L}ip^{\,k} \,\}$,

(5) $\mathcal{L}ip^{\,k}_{\,\mathbb{R}} E = \mathcal{L}ip^{\,k} E \cap (\upsilon_s E)^{\,\mathbb{R}}$,

(6) $\mathcal{L}ip^{\,k}_{\,\mathrm{FKd}} = \mathrm{Con}_{\mathrm{FKd}}{}^{\times 2.} \times \mathbf{U} \cap \{\, (E, F, f) : k \in \infty^+$ and $f \in (\upsilon_s F)^{\,\mathrm{dom}\, f}$
and $\mathrm{dom}\, f \subseteq \upsilon_s E$ and $\forall\, c \in \mathcal{L}ip^{\,k} E\, ;\, f \circ c \in \mathcal{L}ip^{\,k} F \,\}$,

(7) $\mathcal{L}ip^{\,k}_{\,\mathrm{FKa}} = \tau_{\mathrm{dv}} \underset{\mathrm{f}}{\times} \tau_{\mathrm{dv}} \underset{\mathrm{f}}{\times} \mathrm{id}\, ``\, \mathcal{L}ip^{\,k}_{\,\mathrm{FKd}}$.

4 Lemma. *Let* $E, F \in \mathrm{Con}_{\mathrm{FKd}}$, *and with* $G = E \,\boxed{\scriptstyle \mathrm{d}}\, F$, *also let* $c \in (\upsilon_s G)^{\,\mathrm{dom}\, c}$. *Then* $c \in \mathcal{L}ip^{\,\emptyset} G$ *if and only if* $\mathrm{pr}_1 \circ c \in \mathcal{L}ip^{\,\emptyset} E$ *and* $\mathrm{pr}_2 \circ c \in \mathcal{L}ip^{\,\emptyset} F$.

Proof. Put $c_1 = \mathrm{pr}_1 \circ c$ and $c_2 = \mathrm{pr}_2 \circ c$. First letting $c \in \mathcal{L}ip^{\,\emptyset} G$, to have $c_1 \in \mathcal{L}ip^{\,\emptyset} E$, for arbitrarily fixed $\ell_1 \in \tau_{\mathrm{rd}} E$ and $\varphi_1 = \ell_1 \circ c_1$, it suffices that $\varphi_1 \in \mathcal{L}ip^{\,\emptyset} = \mathcal{L}ip$. Letting $\ell_2 = \upsilon_s F \times \{0\}$ and $\ell = \sigma_{\mathrm{rd}}{}^{\mathrm{f}}\mathbb{R} \circ (\ell_1 \underset{\mathrm{f}}{\times} \ell_2)$ and $\varphi = \ell \circ c$, as $\tau_{\mathrm{rd}} F$ is (easily seen to be) a vector subspace in ${}^{\mathrm{f}}\mathbb{R}^{\,\upsilon_s F}]_{\mathrm{vs}}$, we have $\ell_2 \in \tau_{\mathrm{rd}} F$, hence $\ell \in \tau_{\mathrm{rd}} G$, whence by $c \in \mathcal{L}ip^{\,\emptyset} G$ further $\varphi \in \mathcal{L}ip$. For all t having

$$\varphi_1`t = \ell_1 \circ c_1`t + 0 = \sigma_{\mathrm{rd}}{}^{\mathrm{f}}\mathbb{R} \circ (\ell_1 \underset{\mathrm{f}}{\times} \ell_2) \circ [\, c_1, c_2 \,]_{\mathrm{f}}`t = \ell \circ c`t = \varphi`t,$$

we get $\varphi_1 = \varphi \in \mathcal{L}ip$, as we wished. Similarly, one obtains $c_2 \in \mathcal{L}ip^{\,\emptyset} F$.

Conversely, letting $c_1 \in \mathcal{L}ip^{\,\emptyset} E$ and $c_2 \in \mathcal{L}ip^{\,\emptyset} F$, in order to get $c \in \mathcal{L}ip^{\,\emptyset} G$, for arbitrarily fixed $\ell_1 \in \tau_{\mathrm{rd}} E$ and $\ell_2 \in \tau_{\mathrm{rd}} F$, and for $\ell = \sigma_{\mathrm{rd}}{}^{\mathrm{f}}\mathbb{R} \circ (\ell_1 \underset{\mathrm{f}}{\times} \ell_2)$ and $\varphi = \ell \circ c$, it suffices that $\varphi \in \mathcal{L}ip$. Putting $\varphi_\iota = \ell_\iota \circ c_\iota$, then $\varphi_1, \varphi_2 \in \mathcal{L}ip$, and for all t we have

$$\begin{aligned} \varphi`t = \ell \circ c`t &= \sigma_{\mathrm{rd}}{}^{\mathrm{f}}\mathbb{R} \circ (\ell_1 \underset{\mathrm{f}}{\times} \ell_2) \circ [\, c_1, c_2 \,]_{\mathrm{f}}`t = \sigma_{\mathrm{rd}}{}^{\mathrm{f}}\mathbb{R}`(\ell_1 \circ c_1`t, \ell_2 \circ c_2`t) \\ &= \sigma_{\mathrm{rd}}{}^{\mathrm{f}}\mathbb{R}`(\varphi_1`t, \varphi_2`t) = \varphi_1`t + (\varphi_2`t) = (\varphi_1 + \varphi_2)`t, \end{aligned}$$

and hence $\varphi = \varphi_1 + \varphi_2$. Noting that $\{\, u + v : u, v \in \mathcal{L}ip \,\} \subseteq \mathcal{L}ip$, we immediately obtain $\varphi \in \mathcal{L}ip$, as it was required. $\square$

5 Corollary. *For all* E, F, G, f, g, *it holds that*

$$(E, F, f), (E, G, g) \in \mathcal{L}ip^{\,\emptyset}_{\,\mathrm{FKa}} \Rightarrow (E, F \,\boxed{\scriptstyle \mathrm{a}}\, G, [\, f, g \,]_{\mathrm{f}}) \in \mathcal{L}ip^{\,\emptyset}_{\,\mathrm{FKa}}.$$

Proof. In view of Lemma 1, Definitions 2 and (7) and (6) of Constructions 3, for arbitrarily given $(E, F, f), (E, G, g) \in \mathcal{L}ip^{\,\emptyset}_{\,\mathrm{FKd}}$, putting $H = F \,\boxed{\scriptstyle \mathrm{d}}\, G$ and $h = [\, f, g \,]_{\mathrm{f}}$, for an arbitrarily fixed $c \in \mathcal{L}ip^{\,\emptyset} E$ we must show that $h \circ c \in \mathcal{L}ip^{\,\emptyset} H$. Further putting $c_1 = f \circ c$ and $c_2 = g \circ c$, by definition we have $c_1 \in \mathcal{L}ip^{\,\emptyset} F$ and $c_2 \in \mathcal{L}ip^{\,\emptyset} G$. By Lemma 4 for $h \circ c \in \mathcal{L}ip^{\,\emptyset} H$ it suffices that $\mathrm{pr}_1 \circ h \circ c \in \mathcal{L}ip^{\,\emptyset} F$ and $\mathrm{pr}_2 \circ h \circ c \in \mathcal{L}ip^{\,\emptyset} G$ hold. This indeed is the case since we have $\mathrm{pr}_1 \circ h \circ c = \mathrm{pr}_1 \circ [\, f, g \,]_{\mathrm{f}} \circ c = \mathrm{pr}_1 \circ [\, f \circ c, g \circ c \,]_{\mathrm{f}} = \mathrm{pr}_1 \circ [\, c_1, c_2 \,]_{\mathrm{f}} = c_1 \,|\, \mathrm{dom}\, c_2$, and similarly also $\mathrm{pr}_2 \circ h \circ c = c_2 \,|\, \mathrm{dom}\, c_1$. $\square$

6 Proposition. $\mathcal{L}ip^{\,\emptyset}_{\,\mathrm{FKa}}$ *is a* BGN – *class on* $\mathrm{Con}_{\mathrm{FKa}}$ *over* ${}^{\mathrm{tf}}\mathbb{R}$.

Proof. We first note that ${}^{\mathrm{tf}}\mathbb{R} \in \mathrm{Con}_{\mathrm{FKa}}$, and that by [3; Corollary 4.1.7, p. 85] every $\tilde{f} \in \mathcal{L}ip^{\,\emptyset}_{\,\mathrm{FKa}}$ is continuous, trivially having open domain. Hence, for $\mathcal{L}ip^{\,\emptyset}_{\,\mathrm{FKa}}$ to be a productive class on $\mathrm{Con}_{\mathrm{FKa}}$ over ${}^{\mathrm{tf}}\mathbb{R}$ in the sense of [4; Definitions 4, p. 7], for arbitrarily given $E, F \in \mathrm{Con}_{\mathrm{FKa}}$ there must be some $G \in \mathrm{Con}_{\mathrm{FKa}}$ with $\sigma_{\mathrm{rd}} G = (\sigma_{\mathrm{rd}} E) \underset{\mathrm{vs}}{\times} (\sigma_{\mathrm{rd}} F)$ and $(G, E, \mathrm{pr}_1 \,|\, \upsilon_s G), (G, F, \mathrm{pr}_2 \,|\, \upsilon_s G) \in \mathcal{L}ip^{\,\emptyset}_{\,\mathrm{FKa}}$ and such that

$(H,E,f),(H,F,g) \in \mathcal{L}ip^{\emptyset}_{\mathrm{FKa}} \Rightarrow (H,G,[\,f,g\,]_{\mathrm{f}}) \in \mathcal{L}ip^{\emptyset}_{\mathrm{FKa}}$ for all f,g,H. In view of Corollary 5 above, we may take $G = E \boxed{\scriptstyle \mathrm{a}} F$.

In order to establish $(1),\dots(6)$ of [4; Definitions 4, p. 7], below referred to by $(1)_{\mathrm{BGN}},\dots(6)_{\mathrm{BGN}}$, we note the following. We get $(1)_{\mathrm{BGN}}$ from [3; Proposition 4.3.2, p. 99], and $(2)_{\mathrm{BGN}}$ and $(3)_{\mathrm{BGN}}$ follow directly from (6) and (7) of Constructions 3 above. For $(4)_{\mathrm{BGN}}$ with $\iota = \langle\, t^{-1} : t \in \mathbb{R}\setminus\{0\}\,\rangle$, note that we have $|\,\iota`s - \iota`t\,| \le 2\,t^{-2}\,|\,s-t\,|$ when $s,t \in \mathbb{R}$ with $|\,s-t\,| < \frac{1}{2}\,|t| \ne 0$. Using this, an elementary appeal to Constructions 3 with details left to the reader gives $({}^{\mathrm{tf}}\mathbb{R}\,,{}^{\mathrm{tf}}\mathbb{R}\,,\iota) \in \mathcal{L}ip^{\emptyset}_{\mathrm{FKa}}$. For $(5)_{\mathrm{BGN}}$, letting $f,g \in \mathcal{L}ip^{\emptyset}_{\mathrm{FKa}}``\{({}^{\mathrm{tf}}\mathbb{R}\,,F)\}$ hold with $0 \in \mathrm{dom}\, f = \mathrm{dom}\, g$ and $f`t = g`t$ for $t \ne 0$, we should have $f = g$. In order to get this indirectly, supposing that $f \ne g$, we have $f`0 \ne g`0$, and since $\tau_{\mathrm{rd}}(\delta_{\mathrm{av}}`F)$ is point separating, there is $\ell \in \tau_{\mathrm{rd}}(\delta_{\mathrm{av}}`F)$ such that with $\varphi = \ell \circ f$ and $\vartheta = \ell \circ g$ we have $\varphi`0 \ne \vartheta`0$. Since now $\varphi,\vartheta \in \mathcal{L}ip$, there is $\delta \in \mathbb{R}^{+}$ with $\varphi`s \ne \vartheta`s$ for all s with $-\delta < s < \delta$. However, for these $s \ne 0$ we also have $\varphi`s = \ell \circ f`s = \ell \circ g`s = \vartheta`s$, a contradiction.

To get $(6)_{\mathrm{BGN}}$, we have to establish $\tilde{a}, \widetilde{m} \in \mathcal{L}ip^{\emptyset}_{\mathrm{FKa}}$ for an arbitrarily fixed $E \in \mathrm{Con}_{\mathrm{FKa}}$ and $\tilde{a} = (E \boxed{\scriptstyle \mathrm{a}} E, E, \sigma^2_{\mathrm{rd}} E)$ and $\widetilde{m} = (G,E,m)$ where $G = {}^{\mathrm{tf}}\mathbb{R} \boxed{\scriptstyle \mathrm{a}} E$ and $m = \tau\sigma_{\mathrm{rd}} E$. We explicitly show that $\widetilde{m} \in \mathcal{L}ip^{\emptyset}_{\mathrm{FKa}}$, leaving the similar proof of $\tilde{a} \in \mathcal{L}ip^{\emptyset}_{\mathrm{FKa}}$ as an exercise to the reader. Indeed, given any $\Gamma \in \mathcal{L}ip^{\,\emptyset}(\delta_{\mathrm{av}}`G)$, we must get $m \circ \Gamma \in \mathcal{L}ip^{\,\emptyset}(\delta_{\mathrm{av}}`E)$, which in turn follows if for arbitrarily fixed $\ell \in \tau_{\mathrm{rd}}(\delta_{\mathrm{av}}`E)$ we show (m) that $\ell \circ m \circ \Gamma \in \mathcal{L}ip$. Putting $\gamma = \mathrm{pr}_1 \circ \Gamma$ and $c = \mathrm{pr}_2 \circ \Gamma$, by Lemma 4 above we have $\gamma \in \mathcal{L}ip^{\,\emptyset}(\delta_{\mathrm{av}}`{}^{\mathrm{tf}}\mathbb{R}) = \mathcal{L}ip$ and $c \in \mathcal{L}ip^{\,\emptyset}(\delta_{\mathrm{av}}`E)$, and hence $\ell \circ c \in \mathcal{L}ip$. Now, for all t we have

$$\begin{aligned} \ell \circ m \circ \Gamma`t &= \ell \circ m \circ [\,\gamma, c\,]_{\mathrm{f}}`t = \ell`(m`(\gamma`t, c`t))) \\ &= \gamma`t \cdot (\ell`(c`t)) = \gamma`t \cdot (\ell \circ c`t) = \gamma \cdot (\ell \circ c)`t, \qquad \text{whence} \end{aligned}$$

$\ell \circ m \circ \Gamma = \gamma \cdot (\ell \circ c)$. Using $\{\,u \cdot v : u,v \in \mathcal{L}ip\,\} \subseteq \mathcal{L}ip$, we get (m) above. □

7 ***Definitions.*** For all classes $\tilde{f}$ we let

$$\begin{aligned} {}^{\mathrm{a}}\bar{\Delta}_{\mathrm{FK}}\,\tilde{f} &= \mathcal{L}ip^{\emptyset}_{\mathrm{FKa}}\text{-}\bar{\Delta}_{{}^{\mathrm{tf}}\mathbb{R}}\,\tilde{f} \qquad\qquad \text{(see [4; Definitions 7, p. 8])}, \\ {}^{\mathrm{a}}\boldsymbol{\delta}_{\mathrm{FK}}\,\tilde{f} &= \bigcap\{\,(E \boxed{\scriptstyle \mathrm{a}} E, F, g) : \exists\, f, U\,;\ \tilde{f} = (E,F,f) \in \mathrm{Con}_{\mathrm{FKa}}{}^{\times 2.} \times \mathbf{U} \text{ and} \\ &\qquad f \in (\upsilon_s F)^{U} \text{ and } U \in \tau_{\mathrm{rd}} E \text{ and } g = \{\,(x,u,y) : \\ &\qquad\quad \forall\, \varepsilon \in \mathbb{R}^{+}, \ell \in \tau_{\mathrm{rd}}(\delta_{\mathrm{av}}`F)\,;\ \exists\, \delta \in \mathbb{R}^{+}\,;\ \forall\, t \in \mathbb{R}\,; \\ &\qquad\qquad |t| < \delta \Rightarrow |\,\ell`(f`(x + t\,u)_{\mathrm{svs}\,E} - f`x - t\,y)_{\mathrm{svs}\,F}\,| \le |t|\,\varepsilon\,\}\}, \\ {}^{\mathrm{a}}\bar{\Theta}_{\mathrm{FK}}\,\tilde{f} &= \bigcap\{\,\tilde{g} : \exists\, E, F, f, g\,;\ \forall\, h\,;\ \tilde{f} = (E,F,f) \in \mathrm{Con}_{\mathrm{FKa}}{}^{\times 2.} \times \mathbf{U} \text{ and} \\ &\qquad f \in F^{/E} \text{ and } \tilde{g} = (E \boxed{\scriptstyle \mathrm{a}} E \sqcap ({}^{\mathrm{tf}}\mathbb{R} \sqcap {}^{\mathrm{tf}}\mathbb{R}), F, g) \in \mathcal{L}ip^{\emptyset}_{\mathrm{FKa}} \text{ and } [\ h = \\ &\qquad \{\,(x,u\,;s,t\,;y) : f`(x + s\,u)_{\mathrm{svs}\,E} = (f`(x + t\,u)_{\mathrm{svs}\,E} + (s-t)\,y)_{\mathrm{svs}\,F} \ne \mathbf{U}\,\} \\ &\qquad\qquad \Rightarrow g \subseteq h \text{ and } \mathrm{dom}\, h \subseteq \mathrm{dom}\, g\ ]\,\}. \end{aligned}$$

We say that $\tilde{f}$ is *directionally* FKa – *differentiable* if and only if we have

$$\mathrm{dom}\,\tau_{\mathrm{rd}}\,\tilde{f} \times \upsilon_s\sigma^2_{\mathrm{rd}}\,\tilde{f} \subseteq \mathrm{dom}\,\tau_{\mathrm{rd}}{}^{\mathrm{a}}\boldsymbol{\delta}_{\mathrm{FK}}\,\tilde{f} \ne \mathbf{U}.$$

Note above that for example $(x + t\,u)_{\mathrm{svs}\,E} = (\sigma^2_{\mathrm{rd}} E`(x, \tau\sigma_{\mathrm{rd}} E`(t,u)))$ which would usually be denoted by the ambiguous "$(x + t\,u)$", assuming it implicitly understood that the linear structure of the space E is involved in the notation.

In view of [3; Proposition 4.3.12, p. 104] for $\tilde{f} = (E,F,f)$ and $U = \mathrm{dom}\, f$, under the condition that ${}^{\mathrm{a}}\boldsymbol{\delta}_{\mathrm{FK}}\,\tilde{f} \in \mathcal{L}ip^{\emptyset}_{\mathrm{FKa}}$ holds and that $S \subseteq \tau_{\mathrm{rd}}(\delta_{\mathrm{av}}`F)$ is point separating, we have that $\tilde{f}$ is directionally FKa – differentiable if and only if $f : \delta_{\mathrm{av}}`E \supseteq U \to \delta_{\mathrm{av}}`F$ is S – differentiable in the sense [3; Definition 4.3.9, p. 103].

Anyway S – differentiability follows from $\tilde{f}$ being directionally FKa – differentiable by $S \subseteq \tau_{\mathrm{rd}}(\delta_{\mathrm{av}}\grave{}F)$. Then $\tau_{\mathrm{rd}}{}^{\mathrm{a}}\boldsymbol{\delta}_{\mathrm{FK}}\tilde{f}$ is the function denoted by "$\mathrm{d}f$" in [3; Definition 4.3.9, p. 103]. For the beginning of the proof, note the implications

$$\operatorname{dom}\tau_{\mathrm{rd}}{}^{\mathrm{a}}\boldsymbol{\delta}_{\mathrm{FK}}\tilde{f} \neq \mathbf{U} \Rightarrow \tau_{\mathrm{rd}}{}^{\mathrm{a}}\boldsymbol{\delta}_{\mathrm{FK}}\tilde{f} \neq \mathbf{U} \Rightarrow {}^{\mathrm{a}}\boldsymbol{\delta}_{\mathrm{FK}}\tilde{f} \neq \mathbf{U} \Rightarrow \exists\, E, F, f, U\,;$$
$$\tilde{f} = (E, F, f) \in \mathrm{Con}_{\mathrm{FKa}}{}^{\times 2\cdot} \times \mathbf{U} \text{ and } f \in (\upsilon_s F)^U \text{ and } U \in \tau_{\mathrm{rd}} E\,.$$

In [3; p. 103] the map $(\delta_{\mathrm{av}}\grave{}(E \boxdot E), \delta_{\mathrm{av}}\grave{}F, \mathrm{d}f)$ is called the "differential" but following [1; p. 206] we prefer to call ${}^{\mathrm{a}}\boldsymbol{\delta}_{\mathrm{FK}}\tilde{f}$ the *variation*, with the notation $\mathrm{F}\,x = \{(\langle u\rangle, v) : (x, u, v) \in \tau_{\mathrm{rd}}{}^{\mathrm{a}}\boldsymbol{\delta}_{\mathrm{FK}}\tilde{f}\}$ reserving the term differential to referring to the function $g = \langle\, \mathrm{F}\,x : x \in \operatorname{dom}^2 \tau_{\mathrm{rd}}{}^{\mathrm{a}}\boldsymbol{\delta}_{\mathrm{FK}}\tilde{f}\,\rangle$ only in the case where $\mathrm{F}\,x$ is linear $(\sigma_{\mathrm{rd}} E)^{1.]_{\mathrm{vs}}} \to \sigma_{\mathrm{rd}} F$ for every $x \in \operatorname{dom} g$.

8 Lemma. *For every* $\tilde{f}$ *and for* $\tilde{g} = {}^{\mathrm{a}}\bar{\Theta}_{\mathrm{FK}}\tilde{f}$, *either* $\tilde{g} = \mathbf{U}$ *or there are* E, F, f, g *such that* $\tilde{f} = (E, F, f)$ *and* $\tilde{g} = (E \boxdot E \sqcap ({}^{\mathrm{tf}}\mathbb{R} \sqcap {}^{\mathrm{tf}}\mathbb{R}), F, g)$ *with* $\tilde{f}, \tilde{g} \in \mathcal{L}ip^{\emptyset}_{\mathrm{FKa}}$ *and* $\operatorname{dom} g = \{(x, u\,; s, t) : (x + s\,u)_{\mathrm{svs}\,E}, (x + t\,u)_{\mathrm{svs}\,E} \in \operatorname{dom} f\}$, *and*

$$g\grave{}W = ((s - t)^{-1}(f\grave{}(x + s\,u)_{\mathrm{svs}\,E} - f\grave{}(x + t\,u)_{\mathrm{svs}\,E})_{\mathrm{svs}\,F})_{\mathrm{svs}\,F}$$

whenever $W = (x, u\,; s, t) \in \operatorname{dom} g$ *with* $s \neq t$.

Proof. Fix $\tilde{f}$, and suppose that $\tilde{g} \neq \mathbf{U}$. As $\bigcap \emptyset = \mathbf{U}$, then $\{\tilde{g}_1 : (\tilde{g}_1)_{\mathrm{C}}\} \neq \emptyset$, when we let $(\tilde{g}_1)_{\mathrm{C}}$ denote the formula obtained from the four-line formula "$\exists\, E$, $F, f, g\,; \forall\, h\,; \ldots$" occurring in the definition of "${}^{\mathrm{a}}\bar{\Theta}_{\mathrm{FK}}\tilde{f}$" in 7 above by putting there '$\tilde{g}_1$' in place of '$\tilde{g}$'. Hence, there are E, F, f, g such that we have $\tilde{f} = (E, F, f) \in \mathrm{Con}_{\mathrm{FKa}}{}^{\times 2\cdot} \times \mathbf{U}$ and $f \in F^{/E}$ and $\tilde{g}_1 = (E \boxdot E \sqcap ({}^{\mathrm{tf}}\mathbb{R} \sqcap {}^{\mathrm{tf}}\mathbb{R}), F, g) \in \mathcal{L}ip^{\emptyset}_{\mathrm{FKa}}$, and such that letting h be the set of all $(x, u\,; s, t\,; y)$ with the property that $f\grave{}(x + s\,u)_{\mathrm{svs}\,E} = (f\grave{}(x + t\,u)_{\mathrm{svs}\,E} + (s - t)\,y)_{\mathrm{svs}\,F} \neq \mathbf{U}$, then $g \subseteq h$ and $\operatorname{dom} h \subseteq \operatorname{dom} g$. Noting that $\tilde{f}$ uniquely determines E, F, f, if we prove that the preceding conditions also uniquely determine g, we get $\tilde{g} = \tilde{g}_1$, and it remains to establish the formulas for $\operatorname{dom} g$ and $g\grave{}W$, and that also $\tilde{f} \in \mathcal{L}ip^{\emptyset}_{\mathrm{FKa}}$.

Since $f\grave{}(x + s\,u)_{\mathrm{svs}\,E} = (f\grave{}(x + t\,u)_{\mathrm{svs}\,E} + (s - t)\,y)_{\mathrm{svs}\,F} \neq \mathbf{U}$ is equivalent to having $s, t \in \mathbb{R}$ and $x, u \in \upsilon_s E$ and $y \in \upsilon_s F$ such that $x + s\,u, x + t\,u \in \operatorname{dom} f$ and also $f\grave{}(x + s\,u) - f\grave{}(x + t\,u) = (s - t)\,y$ hold, if we put $O = \{(x, u, t) : f\grave{}(x + t\,u)_{\mathrm{svs}\,E} \neq \mathbf{U}\}$ and $Z = \{(x, u\,; t, t) : (x, u, t) \in O\}$, and let h_1 be the function defined by $W \mapsto (s - t)^{-1}(f\grave{}(x + s\,u) - f\grave{}(x + t\,u))$ on the set W of all $W = (x, u\,; s, t)$ with $s \neq t$ and $\{(x, u)\} \times \{s, t\} \subseteq O$, we have $h = \{(x, u\,; t, t) : (x, u, t) \in O\} \times \upsilon_s F \cup h_1$ with $\operatorname{dom} h = W \cup Z$.

From the preceding observations we already get the formulas for $\operatorname{dom} g$ and $g\grave{}W$, provided that g is known to be uniquely determined. To prove this indirectly, supposing that there is another g_1 with the properties of g, there is $W = (x, u\,; t, t) \in Z$ with $g\grave{}W \neq g_1\grave{}W$. Taking $G = E \boxdot E \sqcap ({}^{\mathrm{tf}}\mathbb{R} \sqcap {}^{\mathrm{tf}}\mathbb{R})$ and $\gamma = \langle\, (x, u\,; t + s, t) : s \in \mathbb{R}\,\rangle$, and considering $c = g \circ \gamma$ and $c_1 = g_1 \circ \gamma$, since we have $\gamma \in \mathcal{L}ip^{\,\emptyset} G$, and since $c\grave{}s = c_1\grave{}s$ for $s \neq 0$, by $(5)_{\mathrm{BGN}}$ we get $c = c_1$, and hence in particular $g\grave{}W = g \circ \gamma\grave{}0 = c\grave{}0 = c_1\grave{}0 = g_1 \circ \gamma\grave{}0 = g_1\grave{}W$.

Finally, to get $\tilde{f} \in \mathcal{L}ip^{\emptyset}_{\mathrm{FKa}}$, if $f = \emptyset$ holds, the assertion is trivial directly by definition. Otherwise, we arbitrarily fix any $(x_0, y_0) \in f$, and note that for all x we then have $f\grave{}x = (y_0 + g\grave{}(x, (x - x_0)_{\mathrm{svs}\,E}\,; 0, -1))_{\mathrm{svs}\,F}$. Further, putting $G = E \boxdot E \boxdot ({}^{\mathrm{tf}}\mathbb{R} \boxdot {}^{\mathrm{tf}}\mathbb{R})$ and $\gamma = \langle\, (x, x - x_0\,; 0, -1) : x \in \upsilon_s E\,\rangle$, and using $G = E \boxdot E \sqcap ({}^{\mathrm{tf}}\mathbb{R} \sqcap {}^{\mathrm{tf}}\mathbb{R})$ and $(2)_{\mathrm{BGN}}$ and $(3)_{\mathrm{BGN}}$ and $(6)_{\mathrm{BGN}}$, and either [4; Proposition 6(b), pp. 7–8] or Corollary 5 above, we successively get

$$(E, E \boxdot E, \langle\, (x, -x_0) : x \in \upsilon_s E\,\rangle) \in \mathcal{L}ip^{\emptyset}_{\mathrm{FKa}}\ ,$$

$$(E,E,\langle\, x-x_0 : x\in \upsilon_s E\,\rangle)\in \mathcal{L}ip^{\emptyset}_{\mathrm{FKa}}\ ,$$
$$(E,E \boxdot E,\langle\,(x,x-x_0) : x\in \upsilon_s E\,\rangle)\in \mathcal{L}ip^{\emptyset}_{\mathrm{FKa}}\ ,$$
$$(E,G,\gamma)\in \mathcal{L}ip^{\emptyset}_{\mathrm{FKa}}\ ,\ \ (E,F,g\circ\gamma)\in \mathcal{L}ip^{\emptyset}_{\mathrm{FKa}}\ ,$$
$$(E,F \boxdot F,[\,\upsilon_s E\times\{y_0\},g\circ\gamma\,]_{\mathrm{f}})\in \mathcal{L}ip^{\emptyset}_{\mathrm{FKa}}\ ,\ \ \tilde f\in \mathcal{L}ip^{\emptyset}_{\mathrm{FKa}}\,. \qquad \square$$

In fact, by [3; Corollary 4.5.6, p. 137] in Lemma 8 for all $\tilde f$ we even have either ${}^{\mathrm{a}}\bar\Theta_{\mathrm{FK}}\,\tilde f=\mathbf{U}$ or $\tilde f\in\mathcal{L}ip^{1\cdot}_{\mathrm{FKa}}$. In the case where $\tilde f=(E,F,f)$ with ${}^{\mathrm{a}}\bar\Theta_{\mathrm{FK}}\,\tilde f\neq\mathbf{U}$, in [3; p. 105 ff.] the function $\tau_{\mathrm{rd}}{}^{\mathrm{a}}\bar\Theta_{\mathrm{FK}}\,\tilde f$ is denoted by "ϑf".

Without formulating it as an explicit lemma, we mention that similarly as in the preceding proof, see also [4; Proposition 9, p. 8], one deduces that

for all $\tilde f$ *and for* $\tilde g={}^{\mathrm{a}}\bar\Delta_{\mathrm{FK}}\,\tilde f$, *either* $\tilde g=\mathbf{U}$ *holds, or there are* E,F,f,g *such that* $\tilde f=(E,F,f)$ *and* $\tilde g=(E\boxdot E\sqcap {}^{\mathrm{tf}}\mathbb{R}\,,F,g)$ *with* $\tilde f,\tilde g\in\mathcal{L}ip^{\emptyset}_{\mathrm{FKa}}$ *and* $\operatorname{dom} g=\{\,(x,u,t):(x+t\,u)_{\mathrm{svs}\,E}\in\operatorname{dom} f\,\}$, *and*
$$g\,\grave{}\,z=(t^{-1}\,(f\,\grave{}\,(x+t\,u)_{\mathrm{svs}\,E}-f\,\grave{}\,x)_{\mathrm{svs}\,F})_{\mathrm{svs}\,F}$$
whenever $z=(x,u,t)\in\operatorname{dom} g$ *with* $t\neq 0$.

In the case $\tilde g\neq\mathbf{U}$, by direct appeals to [4; Definitions 7, p. 8] we even see that $\tilde f\in\mathcal{D}^{1\cdot}_{\mathrm{BGN}}(\mathcal{L}ip^{\emptyset}_{\mathrm{FKa}},{}^{\mathrm{tf}}\mathbb{R})$. Using this, in view of Proposition 6 above, from [4; Proposition 10, p. 9] we further get the following

9 Corollary. $\forall\,\tilde f,k\,;\,\tilde f\in\mathcal{D}^{k+1\cdot}_{\mathrm{BGN}}(\mathcal{L}ip^{\emptyset}_{\mathrm{FKa}},{}^{\mathrm{tf}}\mathbb{R})\Leftrightarrow{}^{\mathrm{a}}\bar\Delta_{\mathrm{FK}}\,\tilde f\in\mathcal{D}^{k}_{\mathrm{BGN}}(\mathcal{L}ip^{\emptyset}_{\mathrm{FKa}},{}^{\mathrm{tf}}\mathbb{R})$.

10 Proposition. *For all* $\tilde f,k$ *the implications*

$\tilde f$ *is directionally* FKa*–differentiable and* ${}^{\mathrm{a}}\delta_{\mathrm{FK}}\,\tilde f\in\mathcal{L}ip^{k}_{\mathrm{FKa}}$
$\Rightarrow\ \tilde f\in\mathcal{L}ip^{k+1\cdot}_{\mathrm{FKa}}\ \Rightarrow\ {}^{\mathrm{a}}\bar\Theta_{\mathrm{FK}}\,\tilde f\in\mathcal{L}ip^{k}_{\mathrm{FKa}}$ *hold.*

Proof. In view of the lines 4 – 7 after Definitions 7 above, and also noting that $\mathcal{L}ip^{k}_{\mathrm{FKa}}=\emptyset$ if $k\notin\infty^{+}$, and that $\mathcal{L}ip^{\infty}_{\mathrm{FKa}}=\{\,\tilde f:\forall\,k\in\mathbb{N}_0\,;\,\tilde f\in\mathcal{L}ip^{k}_{\mathrm{FKa}}\,\}$, the asserted implications follow from [3; Definition 4.3.13, Theorem 4.3.24, Corollary 4.5.6, pp. 104 – 106, 110 – 111, 137 – 138]. $\square$

Generally defining
$$\boldsymbol{G}\circ\boldsymbol{F}=\bigcap\{\,(X,Z,g\circ f):\exists\,Y\in\mathbf{U}\,;\,\boldsymbol{F}=(X,Y,f)\text{ and }\boldsymbol{G}=(Y,Z,g)\,\}\,,$$
from [3; Summary 2.4.4 (iii), (6) ⇒ (3), p. 51] we get (1), and directly from (7) and (6) of Constructions 3, noting also Lemma 1 above, we get (2) in the next

11 Proposition. *For all* $E,F,\tilde f,\tilde g,k,\ell$ *it holds that*

(1) $E,F\in\operatorname{Con}_{\mathrm{FKa}}$ *and* $\ell\in\mathcal{L}(E,F)\Rightarrow(E,F,\ell)\in\mathcal{L}ip^{\infty}_{\mathrm{FKa}}$,
(2) $\tilde f,\tilde g\in\mathcal{L}ip^{k}_{\mathrm{FKa}}\Rightarrow\tilde g\circ\tilde f\in\mathcal{L}ip^{k}_{\mathrm{FKa}}$ *or* $\tilde g\circ\tilde f=\mathbf{U}$.

12 Theorem. *The equality* $\mathcal{L}ip^{k}_{\mathrm{FKa}}=\mathcal{D}^{k}_{\mathrm{BGN}}(\mathcal{L}ip^{\emptyset}_{\mathrm{FKa}},{}^{\mathrm{tf}}\mathbb{R})$ *holds for all* k.

Proof. Let $\mathrm{C}^{k}=\mathcal{D}^{k}_{\mathrm{BGN}}(\mathcal{L}ip^{\emptyset}_{\mathrm{FKa}},{}^{\mathrm{tf}}\mathbb{R})$, to simplify the notations a bit. Since we have $\mathrm{D}^{k}=\emptyset$ if $k\notin\infty^{+}$, and since $\mathrm{D}^{\infty}=\{\,\tilde f:\forall\,k\in\mathbb{N}_0\,;\,\tilde f\in\mathrm{D}^{k}\}$ when D^{k} denotes either of $\mathcal{L}ip^{k}_{\mathrm{FKa}}$ and C^{k}, it suffices to prove $\forall\,k\in\mathbb{N}_0\,;\,(k)_{\mathrm{A}}$ by induction, letting $(k)_{\mathrm{A}}$ mean that $\mathcal{L}ip^{k}_{\mathrm{FKa}}=\mathrm{C}^{k}$ holds. We have $(\emptyset)_{\mathrm{A}}$ trivially.

Before considering the inductive step, we note the auxiliary result $(*)$ that $\tilde f$ is directionally FKa–differentiable whenever ${}^{\mathrm{a}}\bar\Delta_{\mathrm{FK}}\,\tilde f\neq\mathbf{U}$ holds. Indeed, then ${}^{\mathrm{a}}\bar\Delta_{\mathrm{FK}}\,\tilde f\in\mathcal{L}ip^{\emptyset}_{\mathrm{FKa}}$ and there are E,F,f with $\tilde f=(E,F,f)$. For arbitrarily fixed $x\in\operatorname{dom} f$ and $u\in\upsilon_s E$, putting $c=\tau_{\mathrm{rd}}{}^{\mathrm{a}}\bar\Delta_{\mathrm{FK}}\,\tilde f\circ\langle\,(x,u,t):t\in\mathbb{R}\,\rangle$, we have $({}^{\mathrm{tf}}\mathbb{R}\,,F,c)\in\mathcal{L}ip^{\emptyset}_{\mathrm{FKa}}$ with $0\in\operatorname{dom} c$ and $t^{-1}(f\,\grave{}\,(x+t\,u)-f\,\grave{}\,x)=c\,\grave{}\,t$ when $0\neq$

$t \in \operatorname{dom} c$. Then having $({}^{\mathrm{tf}}\mathbb{R}\,, {}^{\mathrm{tf}}\mathbb{R}\,, \ell \circ c) \in \mathcal{L}ip^{\,\emptyset}_{\mathrm{FKa}}$ for all $\ell \in \tau_{\mathrm{rd}}\,(\delta_{\mathrm{av}}\grave{}\,F)$, a glance at Definitions 7 shows that we have $(x\,, u\,, c\grave{}\,0) \in \tau_{\mathrm{rd}}{}^{\mathrm{a}}\boldsymbol{\delta}_{\mathrm{FK}}\,\tilde{f}$. From this we see that $\operatorname{dom} \tau_{\mathrm{rd}}{}^{\mathrm{a}}\boldsymbol{\delta}_{\mathrm{FK}}\,\tilde{f} = \operatorname{dom} f \times \upsilon_s E$, hence that $\operatorname{dom} \tau_{\mathrm{rd}}\,\tilde{f} \times \upsilon_s \sigma^2_{\mathrm{rd}}\,\tilde{f} \subseteq \operatorname{dom} \tau_{\mathrm{rd}}{}^{\mathrm{a}}\boldsymbol{\delta}_{\mathrm{FK}}\,\tilde{f} \neq \mathbf{U}$ holds, and consequently that $\tilde{f}$ is directionally FKa$-$differentiable. Note that $\tau_{\mathrm{rd}}{}^{\mathrm{a}}\boldsymbol{\delta}_{\mathrm{FK}}\,\tilde{f}$ is a function since $\tau_{\mathrm{rd}}\,(\delta_{\mathrm{av}}\grave{}\,F)$ is point separating.

Now assuming $(k)_{\mathrm{A}}$ with $k \in \mathbb{N}_0$, to get $(k+1.)_{\mathrm{A}}$, first let $\tilde{f} \in \mathcal{L}ip^{\,k+1.}_{\mathrm{FKa}}$. By Proposition 10 then ${}^{\mathrm{a}}\bar{\Theta}_{\mathrm{FK}}\,\tilde{f} \in \mathcal{L}ip^{\,k}_{\mathrm{FKa}}$. Since for $E = \sigma^2_{\mathrm{rd}}\,\tilde{f}$ and $H = E \boxed{\scriptstyle\mathrm{a}} E \sqcap {}^{\mathrm{tf}}\mathbb{R}$ and $\tilde{\iota}_3 = (H, E \boxed{\scriptstyle\mathrm{a}} E \sqcap ({}^{\mathrm{tf}}\mathbb{R} \sqcap {}^{\mathrm{tf}}\mathbb{R}), \langle\, (x\,, u\,; s\,, 0) : X = (x\,, u\,, s) \in \upsilon_s H\,\rangle)$ we have $\tilde{\iota}_3$ a continuous linear map with ${}^{\mathrm{a}}\bar{\Delta}_{\mathrm{FK}}\,\tilde{f} = {}^{\mathrm{a}}\bar{\Theta}_{\mathrm{FK}}\,\tilde{f} \circ \tilde{\iota}_3 \neq \mathbf{U}$, by $(k)_{\mathrm{A}}$ and Proposition 11 we obtain ${}^{\mathrm{a}}\bar{\Delta}_{\mathrm{FK}}\,\tilde{f} \in \mathcal{L}ip^{\,k}_{\mathrm{FKa}} \subseteq \mathrm{C}^{\,k}$, whence $\tilde{f} \in \mathrm{C}^{\,k+1.}$ follows by Corollary 9 above. Conversely, letting $\tilde{f} \in \mathrm{C}^{\,k+1.}$, by Corollary 9 again we have ${}^{\mathrm{a}}\bar{\Delta}_{\mathrm{FK}}\,\tilde{f} \in \mathrm{C}^{\,k} \subseteq \mathcal{L}ip^{\,k}_{\mathrm{FKa}}$, in view of $(k)_{\mathrm{A}}$. For $\tilde{\iota}_2 = (G, H, \langle\, (x\,, u\,, 0) : X = (x\,, u) \in \upsilon_s G\,\rangle)$ with $G = E \boxed{\scriptstyle\mathrm{a}} E$ having $\tilde{\iota}_2$ a continuous linear map with ${}^{\mathrm{a}}\boldsymbol{\delta}_{\mathrm{FK}}\,\tilde{f} = {}^{\mathrm{a}}\bar{\Delta}_{\mathrm{FK}}\,\tilde{f} \circ \tilde{\iota}_2 \neq \mathbf{U}$, again by Proposition 11 we obtain ${}^{\mathrm{a}}\boldsymbol{\delta}_{\mathrm{FK}}\,\tilde{f} \in \mathcal{L}ip^{\,k}_{\mathrm{FKa}}$. Now $\tilde{f} \in \mathcal{L}ip^{\,k+1.}_{\mathrm{FKa}}$ follows by $(*)$ and Proposition 10 above. □

13 *Remark.* In [4], we used [4; Lemma 50, Corollary 51, pp. 26–27] as an auxiliary tool when establishing [4; Theorem 52] corresponding to Theorem 12 above. Here we did not need to do this kind of work since the corresponding one is already done in sufficient detail in [3; Lemma 4.1.5, Proposition 4.3.11, Theorem 4.5.4, pp. 84, 104, 136–137].

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